Accurate Solutions of the Radiative Transfer Problem via Theory of Connections

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Introduction: Overview

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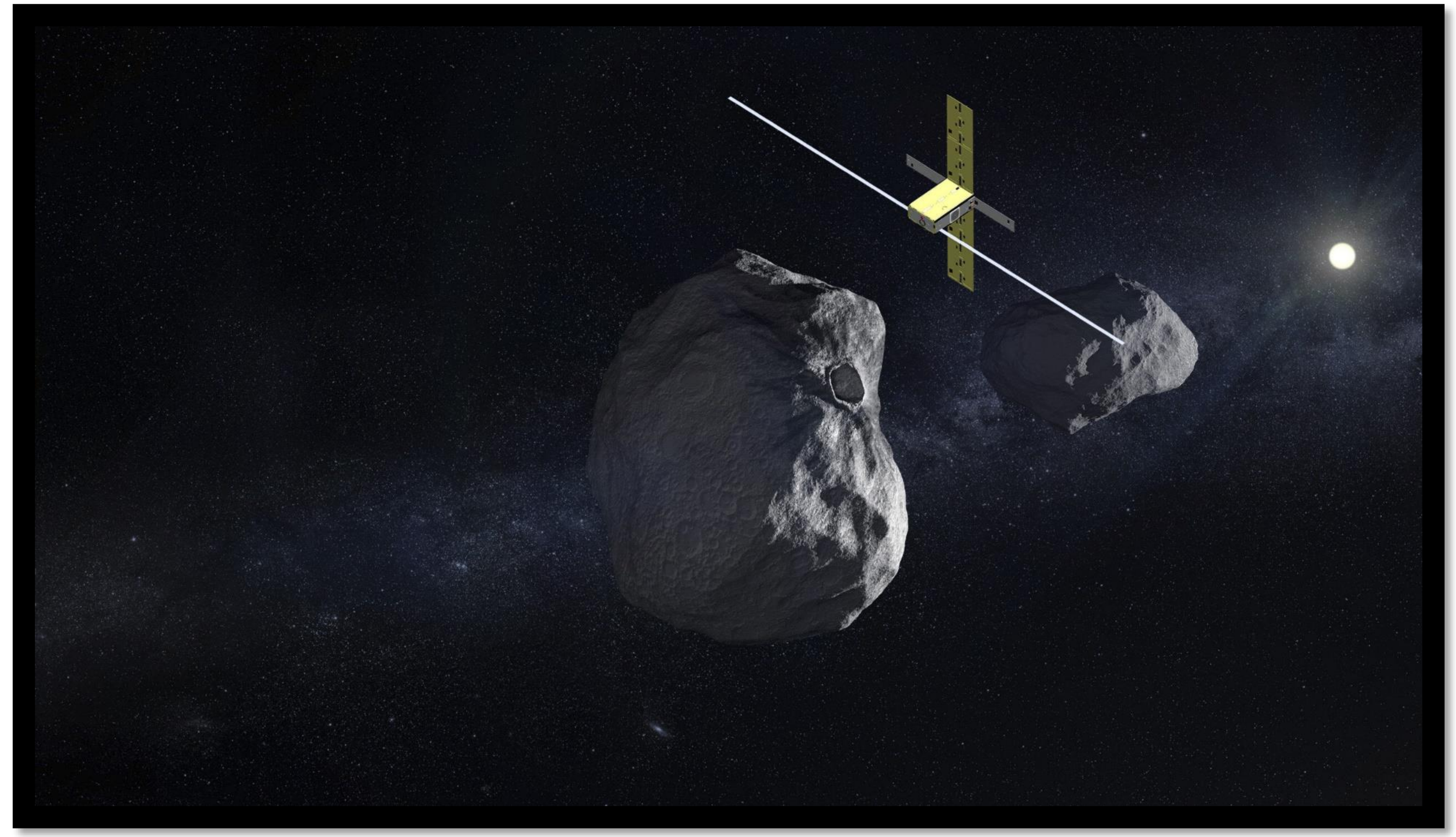
- The *Remote sensing* is the processes of detecting and monitoring an object or an area by measuring its reflected and emitted radiation. It is widely used in the *Planetary Geology* to study surface properties of Planets and Asteroids



- The *Transport Theory* represents the theoretical underpinning of remote sensing. *Radiative Transfer Equation* (RTE) describes how radiation and matter interact based on the particle description of light

Introduction: Goals

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- To solve the RTE using the recently developed *Theory of Connections* (ToC) [Mortari 2018]

- The focus of this talk is to show the capability of ToC in solving 1D Radiative Transfer Equation with high accuracy

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Radiative Transfer for Remote Sensing

- Solving radiative transfer problems for remote sensing is generally hard and computationally expensive
  - No direct analytical solutions except in very limited cases
- Solutions to radiative transfer problems for remote sensing generally are
  - **Semi-analytical**
    - High accuracy in limited cases
  - **Numerical**
    - Hard implementation

ToC approach to solve Linear ODEs

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- ToC derives expressions, called *constrained expressions,* with an embedded set of *n* linear constraints

$$y(t) = g(t) + \sum_{k=1}^{n} \eta_k p_k(t) = g(t) + \boldsymbol{\eta}^T \boldsymbol{p}(t)$$

- According to the literature, the *g(t)* used will be an expansion of orthogonal polynomials (Chebyshev)
  - The solution of the problem is reduced to the calculation of the coefficients of the expansion of Chebyshev polynomials
- ToC has been used to solve several kind of problems, both linear and non-linear, in different areas
  - **Energy Optimal Landing Guidance** – linear- [Furfaro and Mortari 2018];
  - **Fuel Efficient Landing Guidance** – non linear- [Schiassi, Furfaro, et. Al 2019]
    - Machine error accuracy in milliseconds

ToC approach to solve Linear ODEs

Application of the ToC for the solution of a linear ODE, with two constraints at two points:

$$k_2(t)\ddot{y}(t) + k_1(t)\dot{y}(t) + k_0(t)y(t) = f(t) \qquad \text{subject to:} \qquad \begin{cases} y(t_0) = y_0 \\ y(t_f) = y_f \end{cases}$$

- Change of independent variable, to be able to use an expansion of orthogonal polynomials from $t \in [t_0, t_f]$ to $x \in [x_0, x_f]$, where $x_0 = -1$, $x_f = 1$.

The new variable x is defined as:

$$x = c(t - t_0) + x_0$$

Where c is the integration range ratio:

$$c = \frac{x_f - x_0}{t_f - t_0}$$

Due to the derivative chain rule, it follows that:

$$y(t) = y(x) \qquad \frac{dy}{dt} = \dot{y} = c\frac{dy}{dx} = cy' \qquad \frac{d^2y}{dt^2} = \ddot{y} = c^2\frac{d^2y}{dx^2} = c^2y''$$

ToC approach to solve Linear ODEs

- Replacing in the equation we get:

$$c^2 k_2 y''(x) + c k_1 y'(x) + k_0 y(x) = f(x)$$

subject to:

$$\begin{cases} y(x_0) = y_0 \\ y(x_f) = y_f \end{cases}$$

- <u>Constrained expressions</u>

$$y(x) = g(x) + \eta_1 p(x) + \eta_2 q(x)$$

$$y'(x) = g'(x) + \eta_1 p'(x) + \eta_2 q'(x)$$

$$y''(x) = g''(x) + \eta_1 p''(x) + \eta_2 q''(x)$$

where:

$$g(x) = \underline{h}^T(x)\underline{\xi}$$

$$g'(x) = \underline{h}'^T(x)\underline{\xi}$$

$$g''(x) = \underline{h}''^T(x)\underline{\xi}$$

ToC approach to solve Linear ODEs

- Using the boundary conditions, we find η_1, η_2

$$\begin{cases} y_0 = g_0 + \eta_1 p_0 + \eta_2 q_0 \\ y_f = g_f + \eta_1 p_f + \eta_2 q_f \end{cases} \quad \rightarrow \quad \begin{bmatrix} p_0 & q_0 \\ p_f & q_f \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} y_0 - g_0 \\ y_f - g_f \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \boxed{\begin{aligned} \eta_1 &= \frac{1}{\Delta}\left(q_0 \underline{h}_f^T - q_f \underline{h}_0^T\right)\underline{\xi} + \frac{1}{\Delta}\left(q_f y_0 - q_0 y_f\right) \\ \eta_2 &= \frac{1}{\Delta}\left(p_0 \underline{h}_f^T + p_f \underline{h}_0^T\right)\underline{\xi} + \frac{1}{\Delta}\left(p_0 y_f - p_f y_0\right) \end{aligned}}$$

where: $\Delta = p_0 q_f - q_0 p_f \neq 0$

ToC approach to solve Linear ODEs

- By replacing the newly found values of η_1, η_2 in the constrained expressions, we get:

$$y(x) = \left[\underline{h}^T(x) + \frac{p(x)}{\Delta}(q_0\underline{h}_f^T - q_f\underline{h}_0^T) + \frac{q(x)}{\Delta}(q_f\underline{h}_0^T - p_0\underline{h}_f^T)\right]\underline{\xi} + \left[\frac{p(x)}{\Delta}(q_f y_0 - q_0 y_f) + \frac{q(x)}{\Delta}(p_0 y_f - p_f y_0)\right]$$

$$y'(x) = \left[\underline{h}'^T(x) + \frac{p'(x)}{\Delta}(q_0\underline{h}_f^T - q_f\underline{h}_0^T) + \frac{q'(x)}{\Delta}(q_f\underline{h}_0^T - p_0\underline{h}_f^T)\right]\underline{\xi} + \left[\frac{p'(x)}{\Delta}(q_f y_0 - q_0 y_f) + \frac{q'(x)}{\Delta}(p_0 y_f - p_f y_0)\right]$$

$$y''(x) = \left[\underline{h}''^T(x) + \frac{p''(x)}{\Delta}(q_0\underline{h}_f^T - q_f\underline{h}_0^T) + \frac{q''(x)}{\Delta}(q_f\underline{h}_0^T - p_0\underline{h}_f^T)\right]\underline{\xi} + \left[\frac{p''(x)}{\Delta}(q_f y_0 - q_0 y_f) + \frac{q''(x)}{\Delta}(p_0 y_f - p_f y_0)\right]$$

We define the following parameters:

$$\underline{aa} = \frac{(q_0\underline{h}_f^T - q_f\underline{h}_0^T)}{\Delta}$$

$$\underline{bb} = \frac{(q_f\underline{h}_0^T - p_0\underline{h}_f^T)}{\Delta}$$

$$cc = \frac{(q_f y_0 - q_0 y_f)}{\Delta}$$

$$dd = \frac{(p_0 y_f - p_f y_0)}{\Delta}$$

ToC approach to solve Linear ODEs

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Then:

$$y(x) = [\underline{h}^T(x) + p(x)\underline{aa} + q(x)\underline{bb}]\,\underline{\xi} + [p(x)cc + q(x)dd]$$

$$y'(x) = [\underline{h}'^T(x) + p'(x)\underline{aa} + q'(x)\underline{bb}]\,\underline{\xi} + [p'(x)cc + q'(x)dd]$$

$$y''(x) = [\underline{h}''^T(x) + p''(x)\underline{aa} + q''(x)\underline{bb}]\,\underline{\xi} + [p''(x)cc + q''(x)dd]$$

- By plugging into: $c^2 k_2 y''(x) + c k_1 y'(x) + k_0 y(x) = f(x)$

we obtain the equation with the following form:

$$c^2 k_2 \left\{[\underline{h}''^T(x) + p''(x)\underline{aa} + q''(x)\underline{bb}]\,\underline{\xi} + [p''(x)cc + q''(x)dd]\right\} +$$

$$c k_1 \left\{[\underline{h}'^T(x) + p'(x)\underline{aa} + q'(x)\underline{bb}]\,\underline{\xi} + [p'(x)cc + q'(x)dd]\right\} +$$

$$k_0 \left\{[\underline{h}''^T(x) + p''(x)\underline{aa} + q''(x)\underline{bb}]\,\underline{\xi} + [p''(x)cc + q''(x)dd]\right\} = f(x)$$

ToC approach to solve Linear ODEs

By rearranging the terms of the equation just obtained, the solution of the initial ODE is reduced to the solution of a Linear System $A\,\underline{\xi} = \underline{b}$

$$\{A_{ij}\} = \left(c^2 k_2 h_{ij}'' + c k_1 h_{ij}' + k_0 h_{ij}\right) + (c^2 k_2 p_i'' + c k_1 p_i' + k_0 p_i) a a_j + (c^2 k_2 q_i'' + c k_1 q_i' + k_0 q_i) b b_j$$

$$\{b_i\} = f_i - (c^2 k_2 p_i'' + c k_1 p_i' + k_0 p_i) cc - (c^2 k_2 q_i'' + c k_1 q_i' + k_0 q_i) dd$$

- Once we get $\underline{\xi}$ by a Least-Squares, it is replaced in the constrained expressions shown previously:

$$y(x) = \underline{h}^T(x)\underline{\xi} + \eta_1 p(x) + \eta_2 q(x)$$

$$y'(x) = \underline{h}'^T(x)\underline{\xi} + \eta_1 p'(x) + \eta_2 q'(x)$$

$$y''(x) = \underline{h}''^T(x)\underline{\xi} + \eta_1 p''(x) + \eta_2 q''(x)$$

ToC approach to solve Linear ODEs

- We can now reconstruct the solution of the initial equation as a function of t :

$$y(t) = y(x) \qquad \dot{y}(t) = c y'(x) \qquad \ddot{y}(t) = c^2 y''(x)$$

- Finally, by calculating the residuals, we can check the precision of the equation:

$$Res = k_2 \ddot{y}(t) + \mathrm{k}_1 \dot{y}(t) + k_0 y(t) - f(t)$$

PRECISION OF THE EQUATION $\propto \dfrac{\mathbf{1}}{\boldsymbol{Res}}$

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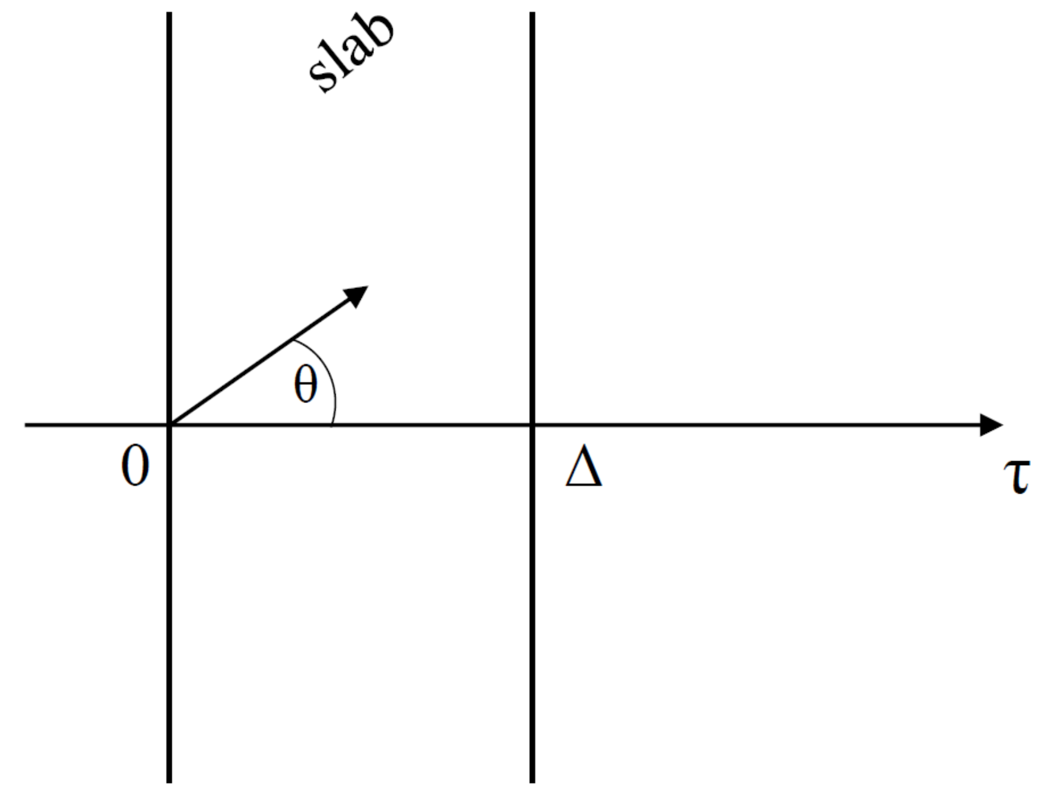
Basic Formulation of the Radiative Transfer Problem

The Radiative Transfer Equation RTE (according to Chandrasekhar) to be solved is:

$$\mu \frac{\partial}{\partial t} I(\tau,\mu,\phi) + I(\tau,\mu,\phi) = \frac{\omega}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} p(\cos\Theta\,)\, I(\tau,\mu',\phi')\,d\phi'\,d\mu'$$



With the following constraints:

$$\begin{cases} I(0,\mu,\phi) = \pi\delta(\mu-\mu_0)\delta(\phi-\phi_0) & \text{for } \mu > 0 \\ I(\Delta,\mu,\phi) = 0 & \text{for } \mu < 0 \end{cases}$$

Radiative Transfer Equation

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- Separation of the Intensity into uncollided fraction and collided fraction (or diffused)

- Making use of the *Addition Theorem of the Spherical Harmonics* to express the phase function:

$$p(\cos\Theta) = \sum_{m=0}^{L}(2-\delta_{0,m})\sum_{l=m}^{L}\beta_l P_l^m(\mu')P_l^m(\mu)\cos[m(\phi'-\phi)]$$

- Expression of the diffused fraction by Fourier series (Siewert 1998):

$$I^*(\tau,\mu,\phi) = \frac{1}{2}\sum_{m=0}^{L}(2-\delta_{0,m})I_m(\tau,\mu)\cos[m(\phi'-\phi)]$$

where the *m*-th Fourier component satisfies the equation of transfer

$$\mu\frac{\partial}{\partial\tau}I_m(\tau,\mu) + I_m(\tau,\mu) = \frac{\omega}{2}\sum_{l=m}^{L}\beta_l P_l^m(\mu)\int_{-1}^{1}P_l^m(\mu')I_m(\tau,\mu')\,d\mu' + \frac{\omega}{2}e^{-\tau/\mu_0}\sum_{l=m}^{L}\beta_l P_l^m(\mu_0)P_l^m(\mu_i)$$

- Discretization of $\mu \rightarrow \mu_i$ where *i = 1, … ,N*

Radiative Transfer Equation

- Change of variable: from τ to x
- Splitting of the equation in forward flux and backward flux.
- Gauss-Legendre quadrature for calculating the integral in the range $[0,1]$.

$$c\mu_i \frac{\partial}{\partial x} I_m^+ + I_m^+ = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) \sum_{k=1}^{N} w_k P_l^m(\mu_k) \left[I_m^+ + (-1)^{l-m} I_m^- \right] + \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(\mu_i)$$

$$-c\mu_i \frac{\partial}{\partial x} I_m^- + I_m^- = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(-\mu_i) \sum_{k=1}^{N} w_k P_l^m(-\mu_k) \left[(-1)^{l-m} I_m^+ + I_m^- \right] + \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(-\mu_i)$$

with following boundary conditions:

$$I_m^+(0) = 0 \qquad I_m^-(\Delta) = 0$$

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Solution of the RTE via ToC: Formulation

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Formulation via ToC

Choise of *p(x)* and *q(x)*
As first Chebyshev polynomials = 1

- Constrained expressions:

$$I_m^+(x) = g^+(x) + \eta^+ p(x) \quad \rightarrow \quad I_m^+(x) = g^+(x) + \eta^+$$

$$I_m^-(x) = g^-(x) + \eta^- q(x) \quad \rightarrow \quad I_m(x)^- = g^-(x) + \eta^-$$

- Use of boundary conditions to find the coefficients η :

$$I_0^+ = 0 = g_0 + \eta^+ \qquad \eta^+ = -g_0$$

$$I_0^- = 0 = g_0 + \eta^- \qquad \eta^- = -g_f$$

- Replacement of η in the constrained expressions:

$$I_m^+(x) = g^+(x) - g_0 \qquad I_m^-(x) = g^-(x) - g_f$$

- Finally, we get the solutions in the following forms:

$$I_m^+ = \left(\boldsymbol{h}^T - \boldsymbol{h}_0^T\right) \cdot \boldsymbol{\xi}^+ \qquad I_m^- = \left(\boldsymbol{h}^T - \boldsymbol{h}_f^T\right) \cdot \boldsymbol{\xi}^-$$

Solution of the RTE via ToC: Formulation

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- Replacement of the constrained expressions in the two DEs

$$(c\mu_i \boldsymbol{h}' + \boldsymbol{h} - \boldsymbol{h_0}) \cdot \boldsymbol{\xi}_i^+ - \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) \sum_{k=1}^{N} w_k P_l^m(\mu_k)\,(\boldsymbol{h} - \boldsymbol{h_0}) \cdot \boldsymbol{\xi}_k^+$$

$$-\frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) \sum_{k=1}^{N} w_k P_l^m(\mu_k)\,(-1)^{l-m} (\boldsymbol{h} - \boldsymbol{h_f}) \cdot \boldsymbol{\xi}_k^- = \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(\mu_i)$$

$$(-c\mu_i \boldsymbol{h}' + \boldsymbol{h} - \boldsymbol{h_f}) \cdot \boldsymbol{\xi}_i^- - \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(-\mu_i) \sum_{k=1}^{N} w_k P_l^m(-\mu_k)\,(\boldsymbol{h} - \boldsymbol{h_f}) \cdot \boldsymbol{\xi}_k^-$$

$$-\frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(-\mu_i) \sum_{k=1}^{N} w_k P_l^m(-\mu_k)\,(-1)^{l-m} (\boldsymbol{h} - \boldsymbol{h_0}) \cdot \boldsymbol{\xi}_k^+ = \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(-\mu_i)$$

- Computation of the coefficients $\boldsymbol{\xi}$ by solving the system $\boldsymbol{A} \cdot \boldsymbol{\xi} = \boldsymbol{b}$

of dimensions: $(2MN \times 2mN) \cdot (2mN \times 1) = (2MN \times 1)$

$\boldsymbol{A}$ $\boldsymbol{\xi}$ $\boldsymbol{b}$

$M =$ spatial discretization points
$N =$ angle discretization points
$m =$ number of polynomials

Solution of the RTE via ToC: Formulation

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$i = 1$, $i = 2$, $i = 3$, $\vdots$, $i = N$

$$\begin{bmatrix}
\blacksquare_1 + \blacksquare_1^1 & \clubsuit_1^1 & \blacksquare_1^2 & \clubsuit_1^2 & \blacksquare_1^3 & \clubsuit_1^3 & \cdots & \cdots & \blacksquare_1^N & \clubsuit_1^N \\
\spadesuit_1^1 & \bullet_1 + \bullet_1^1 & \spadesuit_1^2 & \bullet_1^2 & \spadesuit_1^3 & \bullet_1^3 & \cdots & \cdots & \spadesuit_1^N & \bullet_1^N \\
\blacksquare_2^1 & \clubsuit_2^1 & \blacksquare_2 + \blacksquare_2^2 & \clubsuit_2^2 & \blacksquare_2^3 & \clubsuit_2^3 & \ddots & \ddots & \vdots & \vdots \\
\spadesuit_2^1 & \bullet_2^1 & \spadesuit_2^2 & \bullet_2 + \bullet_2^2 & \spadesuit_2^3 & \bullet_2^3 & \ddots & \ddots & \vdots & \vdots \\
\blacksquare_3^1 & \clubsuit_3^1 & \blacksquare_3^2 & \clubsuit_3^2 & \blacksquare_3 + \blacksquare_3^3 & \clubsuit_3^3 & \ddots & \ddots & \vdots & \vdots \\
\spadesuit_3^1 & \bullet_3^1 & \spadesuit_3^2 & \bullet_3^2 & \spadesuit_3^3 & \bullet_3 + \bullet_3^3 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\blacksquare_N^1 & \clubsuit_N^1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \blacksquare_N + \blacksquare_N^N & \clubsuit_N^N \\
\spadesuit_N^1 & \bullet_N^1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \spadesuit_N^N & \bullet_N + \bullet_N^N
\end{bmatrix} \cdot \begin{bmatrix} \xi_1^+ \\ \xi_1^- \\ \xi_2^+ \\ \xi_2^- \\ \xi_3^+ \\ \xi_3^- \\ \vdots \\ \vdots \\ \vdots \\ \xi_N^+ \\ \xi_N^- \end{bmatrix} = \begin{bmatrix} b_1^+ \\ b_1^- \\ b_2^+ \\ b_2^- \\ b_3^+ \\ b_3^- \\ \vdots \\ \vdots \\ \vdots \\ b_N^+ \\ b_N^- \end{bmatrix}$$

$k = 1$ $\quad k = 2$ $\quad k = 3$ $\quad \cdots \cdots$ $\quad k = N$

where:

$$\blacksquare_i = c\mu_i \boldsymbol{h}' + \boldsymbol{h} - \boldsymbol{h_0} \qquad \bullet_i = -c\mu_i \boldsymbol{h}' + \boldsymbol{h} - \boldsymbol{h_f}$$

$$\blacksquare_i^k = -\frac{\omega}{2} w_k (\boldsymbol{h} - \boldsymbol{h_0}) \sum_{l=m}^{L} \beta_l P_l(\mu_i) P_l(\mu_k); \qquad \clubsuit_i^k = -\frac{\omega}{2} w_k (\boldsymbol{h} - \boldsymbol{h_f}) \sum_{l=m}^{L} \beta_l P_l(\mu_i) P_l(\mu_k) (-1)^{l-m}$$

$$\spadesuit_i^k = -\frac{\omega}{2} w_k (\boldsymbol{h} - \boldsymbol{h_0}) \sum_{l=m}^{L} \beta_l P_l(-\mu_i) P_l(-\mu_k) (-1)^{l-m}; \qquad \bullet_i^k = -\frac{\omega}{2} w_k (\boldsymbol{h} - \boldsymbol{h_f}) \sum_{l=m}^{L} \beta_l P_l(-\mu_i) P_l(-\mu_k)$$

Solution of the RTE via ToC: Formulation

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- Substitution of ξ coefficients in the constrained expressions

$$\boldsymbol{I_m^+} = (\boldsymbol{h} - \boldsymbol{h_0}) \cdot \boldsymbol{\xi}^+ \qquad \boldsymbol{I_m^-} = (\boldsymbol{h} - \boldsymbol{h_f}) \cdot \boldsymbol{\xi}^-$$

- Substitution of the *m-th* Fourier series component:

$$I_*^+(\tau,\mu,\phi) = \frac{1}{2}\sum_{m=0}^{L}(2-\delta_{0,m})\boldsymbol{I_m^+}(\boldsymbol{\tau},\boldsymbol{\mu})\cos[m(\phi'-\phi)] \qquad I_*^-(\tau,\mu,\phi) = \frac{1}{2}\sum_{m=0}^{L}(2-\delta_{0,m})\boldsymbol{I_m^-}(\boldsymbol{\tau},\boldsymbol{\mu})\cos[m(\phi'-\phi)]$$

- *Post-processing,* to find solutions at every polar angle, and at any slab's point (via ToC)

$$(c\gamma_j\boldsymbol{h}' + \boldsymbol{h} - \boldsymbol{h_0}) \cdot \boldsymbol{\zeta_j^+} = \frac{\omega}{2}\sum_{l=m}^{L}\beta_l P_l^m(\gamma_j)\sum_{k=1}^{N} w_k P_l^m(\mu_k)\left[(\boldsymbol{h} - \boldsymbol{h_0}) \cdot \xi_k^+ + (-1)^{l-m}(\boldsymbol{h} - \boldsymbol{h_f}) \cdot \xi_k^-\right] + b_j^+$$

$$(-c\gamma_j\boldsymbol{h}' + \boldsymbol{h} - \boldsymbol{h_f}) \cdot \boldsymbol{\zeta_j^-} = \frac{\omega}{2}\sum_{l=m}^{L}\beta_l P_l^m(-\gamma_j)\sum_{k=1}^{N} w_k P_l^m(-\mu_k)\left[(-1)^{l-m}(\boldsymbol{h} - \boldsymbol{h_0}) \cdot \xi_k^+ + (\boldsymbol{h} - \boldsymbol{h_f}) \cdot \xi_k^-\right] + b_j^-$$

The new arbitrary angles are γ_j and the new unknown vector is $\zeta_j^{\pm}$, computed by a Least-Squares method.

Solution of the RTE via ToC: Results

- The accuracy of this new method for the RTE solution was validated by comparing the results with the benchmarks published by C.E. Siewert et. al, for the following case studies:
  - Isotropic, Two-Stream Approximation;
  - Isotropic, Multi-stream;
  - Anisotropic, Mie Scattering;
  - Anisotropic, Haze L Problem.

- For all the cases considered, we obtained the same digits published by Garcia & Siewert (1986)

Solution of the RTE via ToC: Results

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Haze L Problem

for *m=0* Fourier component (normal incident beam and conservative case)

$$\omega = 1$$
$$\mu_0 = 1$$
$$\Delta = 1$$

RTE via ToC
N = 30

vs.

RTE via ADO
N = 100 ÷150

- CPU time for the Least-Squares ≅ 8,5 seconds
- Total CPU time to run the code (including matrices construction, plots and post-processing) ≅ 25 seconds

Table 11: Haze L Problem - The Intensity $I_*(\tau,\mu)$ for the Haze L phase function with $\omega = 1$ and $\mu_0 = 1$.

μ	$\tau = 0$	$\tau = 0.05\Delta$	$\tau = 0.1\Delta$	$\tau = 0.2\Delta$	$\tau = 0.5\Delta$	$\tau = 0.75\Delta$	$\tau = \Delta$
-1.0	3.61452e-2	3.43394e-2	3.25109e-2	2.88122e-2	1.76286e-2	8.52589e-3	**0**
-0.9	3.97819e-2	3.78723e-2	3.59207e-2	3.19303e-2	1.96202e-2	9.45731e-3	**0**
-0.8	4.27313e-2	4.08406e-2	3.88734e-2	3.47677e-2	2.16019e-2	1.03959e-2	**0**
-0.7	4.80051e-2	4.61319e-2	4.41307e-2	3.98292e-2	2.52479e-2	1.22171e-2	**0**
-0.6	5.58214e-2	5.40432e-2	5.20594e-2	4.75986e-2	3.11837e-2	1.53618e-2	**0**
-0.5	6.60942e-2	6.46296e-2	6.28449e-2	5.84971e-2	4.02740e-2	2.05621e-2	**0**
-0.4	7.81481e-2	7.74403e-2	7.62508e-2	7.27049e-2	5.37300e-2	2.91285e-2	**0**
-0.3	8.99682e-2	9.07706e-2	9.08784e-2	8.94711e-2	7.29643e-2	4.34688e-2	**0**
-0.2	9.70815e-2	1.00421e-1	1.02789e-1	1.05506e-1	9.83777e-2	6.79949e-2	**0**
-0.1	9.29328e-2	9.98187e-2	1.05195e-1	1.13497e-1	1.24037e-1	1.08399e-2	**0**
-0.0	6.98774e-2	8.46673e-2	9.41663e-1	1.08727e-1	1.35762e-1	1.42779e-1	**0**
0.0	**0**	8.46673e-2	9.41663e-1	1.08727e-1	1.35762e-1	1.42779e-1	1.14808e-1
0.1	**0**	2.95418e-2	5.24346e-2	8.45649e-2	1.35096e-1	1.56106e-1	1.56976-1
0.2	**0**	1.64907e-2	3.22817e-2	6.07527e-2	1.24350e-1	1.58925e-1	1.76818e-1
0.3	**0**	1.23421e-2	2.48488e-2	4.93968e-2	1.14811e-1	1.57937e-1	1.88301e-1
0.4	**0**	1.11879e-2	2.26450e-2	4.57547e-2	1.12269e-1	1.60862e-1	2.00019e-1
0.5	**0**	1.17959e-2	2.37910e-2	4.80003e-2	1.19079e-1	1.73191e-1	2.19633e-1
0.6	**0**	1.42049e-2	2.84584e-2	5.68731e-2	1.39051e-1	2.01445e-1	2.55983e-1
0.7	**0**	1.95833e-2	3.89248e-2	7.67454e-2	1.82004e-1	2.58986e-1	3.25125e-1
0.8	**0**	3.19532e-2	6.29430e-2	1.22045e-1	2.77182e-1	3.82767e-1	4.68658e-1
0.9	**0**	6.87267e-2	1.33917e-1	2.54259e-1	5.44601e-1	7.19447e-1	8.46084e-1
1.0	**0**	3.64940e-1	7.00266e-1	1.28955	2.52255	3.09319	3.38091

Solution of the RTE via ToC: Results

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Haze L Problem

for 83 *m* Fourier components

$$\omega = 0.9$$
$$\mu_0 = 0.5$$
$$\Delta = 1$$
$$\phi - \phi_0 = \pi/2$$

RTE via ToC
N = 30

vs.

RTE via ADO
N = 100 ÷150

- CPU time per Least-Squares for each $m \cong 8{,}5$ seconds
- Total CPU time for any Least-Squares $\cong 11{,}7$ minutes
- Total CPU time to run the code (including matrices construction, plots and post-processing) $\cong 25$ minutes

Table 13: Haze L Problem - The Intensity $I_*(\tau, \mu, \phi)$ for the Haze L phase function with $\omega = 0.9$, $\mu_0 = 0.5$, and $\phi - \phi_0 = \pi/2$.

μ	$\tau = 0$	$\tau = 0.05\Delta$	$\tau = 0.1\Delta$	$\tau = 0.2\Delta$	$\tau = 0.5\Delta$	$\tau = 0.75\Delta$	$\tau = \Delta$
-1.0	2.28190e-2	2.14170e-2	1.99920e-2	1.71574e-2	9.34719e-3	4.02513e-3	0
-0.9	2.69861e-2	2.54001e-2	2.3770e-2	2.04885e-2	1.12507e-2	4.83998e-3	0
-0.8	3.23251e-2	3.05433e-2	2.86841e-2	2.48816e-2	1.38576e-2	5.98687e-3	0
-0.7	3.90915e-2	3.71288e-2	3.50364e-2	3.06624e-2	1.74617e-2	7.63435e-3	0
-0.6	4.75194e-2	4.54446e-2	4.31587e-2	3.82274e-2	2.24929e-2	1.00585e-2	0
-0.5	5.76960e-2	5.56800e-2	5.33274e-2	4.79966e-2	2.95696e-2	1.37243e-2	0
-0.4	6.92921e-2	6.76843e-2	6.55506e-2	6.02592e-2	3.95485e-2	1.94423e-2	0
-0.3	8.09723e-2	8.04082e-2	7.90373e-2	7.47154e-2	5.34553e-2	2.86762e-2	0
-0.2	8.94088e-2	9.08993e-2	9.11597e-2	8.93864e-2	7.18225e-2	4.41114e-2	0
-0.1	8.86327e-2	9.36078e-2	9.65669e-2	9.91642e-2	9.15413e-2	6.94491e-2	0
-0.0	6.76014e-2	8.16018e-2	8.92220e-2	9.83762e-2	1.03484e-1	9.32369e-2	0
0.0	0	8.16018e-2	8.92220e-2	9.83762e-2	1.03484e-1	9.32371e-2	6.29164e-2
0.1	0	2.74475e-2	4.83619e-2	7.57571e-2	1.04622e-1	1.04387e-1	8.95907e-2
0.2	0	1.41330e-2	2.75945e-2	5.09061e-2	9.28868e-2	1.04678e-1	1.01178e-1
0.3	0	9.26644e-3	1.87294e-2	3.68737e-2	7.85470e-2	9.74004e-2	1.02990e-1
0.4	0	6.96644e-3	1.42411e-2	2.88244e-2	6.68034e-2	8.82947e-2	9.95180e-2
0.5	0	5.75720e-3	1.17847e-2	2.40718e-2	5.82338e-2	8.01154e-2	9.43192e-2
0.6	0	5.11030e-3	1.04251e-2	2.12819e-2	5.23831e-2	7.37544e-2	8.93298e-2
0.7	0	4.79703e-3	9.73440e-3	1.97631e-2	4.87196e-2	6.93677e-2	8.54626e-2
0.8	0	4.71115e-3	9.50506e-3	1.91501e-2	4.68396e-2	6.68825e-2	8.31200e-2
0.9	0	4.80640e-3	9.64244e-3	1.92635e-2	4.64998e-2	6.62130e-2	8.24990e-2
1.0	0	5.07113e-3	1.01191e-2	2.00435e-2	4.76083e-2	6.73492e-2	8.37579e-2

Contents

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Conclusions and Outlooks

- The RTE is solved via the recently developed ToC
  - The accuracy of the results is compared with the recognized benchmarks
  - Straightforward implementation
  - Reformulation not required for the conservative case $\omega=1$

- Future developments
  - To use this new methodology to compute the *Reflectance*, for the study of asteroid binary systems properties through light-curves inversions
  - To solve RTE for the multi-slab case
  - To solve the 3D time-dependent RTE

Thanks for the attention

Questions time

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